

From $N=2$ Fermionic Strings to Superstrings?^{*+}

Olaf Lechtenfeld

*Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
E-mail: lechtenf@itp.uni-hannover.de*

Abstract

I review the covariant quantization of the critical $N=2$ fermionic string with and without a global \mathbf{Z}_2 twist. The BRST analysis yields massless bosonic and fermionic vertex operators in various ghost and picture number sectors, as well as picture-changers and their inverses, depending on the field basis chosen for bosonization. Two distinct GSO projections exist, one (untwisted) retaining merely the known bosonic scalar and its spectral-flow partner, the other (twisted) yielding two fermions and one boson, on the massless level. The absence of interactions in the latter case rules out standard spacetime supersymmetry. In the untwisted theory, the $U(1,1)$ -invariant three-point and vanishing four-point functions are confirmed at tree level. I comment on the $N=2$ string field theory, the integration over moduli and the realization of spectral flow.

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In this talk I am reporting on some recent progress in unraveling the BRST structure of the critical $N=2$ fermionic string. This investigation has been a joint effort including Jan Bischoff, a student of mine, Sergei Ketov, postdoc at Hannover, as well as Andrew Parkes, now at Edinburgh. For more details, I refer to our preprints [1, 2].

There are a number of reasons why the $N=2$ extended fermionic string [3, 4] is worth studying in spite of the fact that its critical real dimension equals $4+0$ or $2+2$, excluding $D = 1+3$ [5, 6]. First, there is the curious fact that this object is actually just a point particle pretending to be a string, since it has only a finite number of physical excitations [7]. Second, it is natural to wonder if the known connection between world-sheet and spacetime supersymmetry extends to this case, as has been advocated by Siegel [8]. Third, $N=2$ strings are known to be closely related with self-dual four-dimensional field theories and integrable models [9]. Fourth, the $N=0$ (bosonic) and $N=1$ (fermionic) strings can also be considered in the framework of the general $N=2$ string theory as particular vacua [10, 11, 12, 13]. Fifth, the moduli spaces of $N=2$ super Riemann surfaces carry novel structure due to the presence of an additional $U(1)$ gauge field. Finally, the mere existence of this string theory warrants its study. For a review of the subject, consult refs. [14, 15].

We begin by reviewing the BRST quantization of the $N=2$ string and add new results concerning possible twisting, chiral bosonization, and picture-changing and its inverse, as we go along. Our starting point is the $N=2$ world-sheet supergravity action [16]. The extended supergravity multiplet involves the real zweibein e_α^a , a *complex* gravitino χ_α and a real abelian gauge field (graviphoton) A_α , with $\alpha = 0, 1$. The conformal matter consists of two complex string coordinates Z^μ and two complex spin $1/2$ (NSR) fermions ψ^μ , with $\mu = 0, 1$. In this talk, I choose the signature of the real world-sheet metric and the complex spacetime metric both to be $(-+)$.

When decomposing the complex matter degrees of freedom into real ones, two different field bases suggest themselves: A ‘real basis’ $Z^{i\mu}$, $i = 2, 3$, should read

$$(Z^{2\mu}, Z^{3\mu}) := (\text{Re}Z^\mu, \text{Im}Z^\mu) \quad , \quad (1a)$$

whereas a ‘holomorphic basis’ would be

$$(Z^{+\mu}, Z^{-\mu}) := (Z^\mu, Z^{*\mu}) \quad . \quad (1b)$$

The unusual range of the index i helps to avoid confusing the same numerical values of μ and i .

The coordinates Z take values in the target space $\mathbf{C}^{1,1}$. Besides the local $N=2$ superconformal world-sheet symmetry, the action also has a global spacetime symmetry

given by rigid translations plus ‘Lorentz transformations’ comprising $U(1, 1) \times \mathbf{Z}_2$ [9]. The \mathbf{Z}_2 factor amounts to complex conjugation. The string can be twisted by identifying fields upon complex conjugation, e.g. $Z^\mu \sim Z^{*\mu}$. In the real basis, this effectively puts $Z^{3\mu} = 0$ and changes the target to the half-spacetime $\mathbf{C}^{1,1}/\mathbf{Z}_2$.

Transporting the matter fields along a non-contractible cycle on the world-sheet, the NSR fermions $\psi^{i\mu}$ show a \mathbf{Z}_2 monodromy, since they live in a spin bundle.¹ For the bosonic coordinates $Z^{i\mu}$ the monodromy group is either trivial (untwisted theory) or again \mathbf{Z}_2 (twisted case), depending on whether or not we allow $Z^{3\mu}$ to be antiperiodic. ‘Lorentz invariance’ demands the monodromies to be independent of the μ index. Moreover, single-valuedness of the Brink-Schwarz lagrangian requires that the product of the ($i=2$) and ($i=3$) monodromies be the same for Z and for ψ . Thus, one ends up with four sectors:

sector	$Z^{2\mu}$	$Z^{3\mu}$	$\psi^{2\mu}$	$\psi^{3\mu}$
(NS,NS)	P	P	P	P
(R,R)	P	P	A	A
(NS,R)	P	A	P	A
(R,NS)	P	A	A	P

Here, P and A refer to periodic and antiperiodic boundary conditions, respectively, in natural coordinates on the cylinder. The untwisted string consists of the first two sectors only, which then are connected by spectral flow [9]. The twisted theory contains all four sectors and was first considered by Mathur and Mukhi [17]. As a consequence of the twist, however, the graviphoton A_α becomes antiperiodic and, hence, must be set to zero. This means that the $N=2$ world-sheet supersymmetry gets broken to $N=1$ by the \mathbf{Z}_2 twist, and the spectral flow disappears in this case. More general monodromies from $U(1, 1)$ are compatible with the action, but have been shown not to lead to massless physical states (with one curious exception) [1].

Via BRST quantization in the $N=2$ superconformal gauge the fields (and ghosts) of the $N=2$ string on the euclidean world-sheet become free, so that they can be decomposed into their holomorphic and anti-holomorphic parts, as is usual in 2d conformal field theory. From now on, our notation will refer to the left-moving (holomorphic) part of the conformal fields. The ghost systems appropriate for the $N=2$ string are:

¹ In the untwisted case, the monodromy group is actually $U(1)$ due to the spectral flow in the $N=2$ superconformal algebra. We shall comment on this later.

- the reparametrization ghosts (b, c) , an anticommuting pair of free world-sheet fermions with conformal dimensions $(2, -1)$.
- the two-dimensional $N=2$ supersymmetry ghosts (β^i, γ^i) or (β^\mp, γ^\pm) , two commuting pairs of free world-sheet fermions with conformal dimensions $(\frac{3}{2}, -\frac{1}{2})$.
- the $U(1)$ ghosts (\tilde{b}, \tilde{c}) , an anticommuting pair of free world-sheet fermions with conformal dimensions $(1, 0)$.

The $N=2$ string BRST charge

$$Q_{\text{BRST}} = \oint_0 \frac{dz}{2\pi i} j_{\text{BRST}}(z) \quad (2)$$

is expressed through the dimension-one BRST current which in its most convenient form reads

$$j_{\text{BRST}} = c\hat{T} + bc\partial c + \gamma^2 G + \gamma^3 \overline{G} + \tilde{c}\hat{J} \\ - (\gamma^2 \gamma^2 + \gamma^3 \gamma^3)b + 2i(\gamma^2 \partial \gamma^3 - \gamma^3 \partial \gamma^2)\tilde{b} + \frac{3}{4}\partial[c(\beta^2 \gamma^2 + \beta^3 \gamma^3)] \quad (3a)$$

in the real basis, or

$$j_{\text{BRST}} = c\hat{T} + bc\partial c + \frac{1}{2}\gamma^- G^+ + \frac{1}{2}\gamma^+ G^- + \tilde{c}\hat{J} \\ - \gamma^+ \gamma^- b + (\gamma^- \partial \gamma^+ - \gamma^+ \partial \gamma^-)\tilde{b} + \frac{3}{8}\partial[c(\beta^+ \gamma^- + \beta^- \gamma^+)] \quad (3b)$$

in its holomorphic form. Here, we use the notation

$$\hat{T} = T_{\text{tot}} - T_{b,c} \quad \text{and} \quad \hat{J} = J_{\text{tot}} - \partial(\tilde{b}c) , \quad (4)$$

where $T_{b,c} = -2b\partial c - (\partial b)c$, and we introduced the full (BRST-invariant) stress tensor T_{tot} and the $U(1)$ current J_{tot} as

$$T_{\text{tot}} = \{Q_{\text{BRST}}, b\} = T + T_{b,c} - \tilde{b}\partial\tilde{c} - \frac{3}{2}(\beta^2 \partial \gamma^2 + \beta^3 \partial \gamma^3) - \frac{1}{2}(\gamma^2 \partial \beta^2 + \gamma^3 \partial \beta^3) \\ = T + T_{b,c} - \tilde{b}\partial\tilde{c} - \frac{3}{4}(\beta^+ \partial \gamma^- + \beta^- \partial \gamma^+) - \frac{1}{4}(\gamma^+ \partial \beta^- + \gamma^- \partial \beta^+) , \\ J_{\text{tot}} = \{Q_{\text{BRST}}, \tilde{b}\} = J + \partial(\tilde{b}c) - \frac{i}{2}(\beta^2 \gamma^3 - \beta^3 \gamma^2) \\ = J + \partial(\tilde{b}c) + \frac{1}{4}(\beta^+ \gamma^- - \beta^- \gamma^+) . \quad (5)$$

Above, T , G and J are the $N=2$ string (matter) currents without ghosts, *viz.*

$$T = -\frac{1}{2}(\partial Z^i \cdot \partial Z^i - \psi^i \cdot \partial \psi^i) = -\frac{1}{2}\partial Z^+ \cdot \partial Z^- + \frac{1}{4}\psi^+ \cdot \partial \psi^- + \frac{1}{4}\psi^- \cdot \partial \psi^+ , \\ G = \delta^{ij}\partial Z^i \cdot \psi^j , \quad G^+ = \partial Z^- \cdot \psi^+ , \\ \overline{G} = \varepsilon^{ij}\partial Z^i \cdot \psi^j , \quad G^- = \partial Z^+ \cdot \psi^- , \\ J = \frac{i}{4}\varepsilon^{ij}\psi^i \cdot \psi^j = \frac{i}{2}\psi^2 \cdot \psi^3 = -\frac{1}{4}\psi^+ \cdot \psi^- . \quad (6)$$

In order to construct fermionic vertex operators we make use of chiral bosonization [18, 19]. This requires us to switch to a Cartan-Weyl or light-cone basis with respect to the index μ , e.g.

$$\psi^{\pm} = \psi^0 \pm \psi^1 \quad . \quad (7)$$

For the complex fields ψ , β , and γ , bosonization depends on the basis. As for the $N=1$ string, the Fock space of the first-order fields gets imbedded into an extended Fock space of second-order bosonic fields,

$$\begin{aligned} \text{real basis :} \quad & \psi^{i+}, \psi^{i-}; \beta^i, \gamma^i \quad \Longrightarrow \quad \phi^i; \varphi^i, \eta^i, \partial\xi^i \\ \text{holomorphic basis :} \quad & \psi^{\pm+}, \psi^{\pm-}; \beta^{\pm}, \gamma^{\pm} \quad \Longrightarrow \quad \phi^{\pm}; \varphi^{\pm}, \eta^{\pm}, \partial\xi^{\pm} \quad , \end{aligned} \quad (8)$$

where we chose not to bosonize the auxiliary fermionic (η, ξ) system. The creation of Ramond states out of the Neveu-Schwarz vacuum requires the use of matter and ghost spin fields, which are constructed as $e^{\pm\frac{1}{2}\phi}$ and $e^{\pm\frac{1}{2}\varphi}$, respectively. Furthermore, matter and ghost twist fields are needed to generate twisted states. It should be stressed that the relation between the two bosonization schemes is non-local and non-polynomial, although the unbosonized field bases are linearly related. This is exemplified by

$$\begin{aligned} \hat{J} &= \frac{i}{2}\psi^2 \cdot \psi^3 + \frac{i}{2}(\gamma^2\beta^3 - \gamma^3\beta^2) = -\frac{i}{2}e^{+\phi^2-\phi^3} + \frac{i}{2}e^{+\varphi^2-\varphi^3}\eta^2\partial\xi^3 - (2 \leftrightarrow 3) \\ &= -\frac{1}{4}\psi^+ \cdot \psi^- - \frac{1}{4}(\gamma^+\beta^- - \gamma^-\beta^+) = \frac{1}{2}(\partial\phi^+ - \partial\phi^-) + \frac{1}{2}(\partial\varphi^+ - \partial\varphi^-) \quad . \end{aligned} \quad (9)$$

The BRST cohomology problem is simplified by identifying grading operators. For the $N=2$ string, these are

- the total ghost charge $U = -\oint[bc + \tilde{b}\tilde{c} + \frac{1}{2}\beta^+\gamma^- + \frac{1}{2}\beta^-\gamma^+]$
- the picture charges $\Pi^i = -\oint[\beta^i\gamma^i + \eta^i\xi^i] \quad i = 2 \text{ or } 3$
or $\Pi^{\pm} = -\frac{1}{2}\oint[\beta^{\pm}\gamma^{\mp} + \eta^{\pm}\xi^{\mp}] \quad + \text{ or } -$
- the full bosonic constraints T_{tot} and J_{tot}

of which only U does not commute with Q_{BRST} . Accordingly, it suffices to separately investigate simultaneous eigenspaces of the commuting set $\{U, \Pi, L_0^{\text{tot}}, J_0^{\text{tot}}\}$, labelled by $\{u, \pi, h, e\}$. Note that the picture charges are basis-dependent. From $L_0^{\text{tot}} = \{Q_{\text{BRST}}, b_0\}$ and $J_0^{\text{tot}} = \{Q_{\text{BRST}}, \tilde{b}_0\}$ it readily follows that non-trivial cohomology only exists for $h = e = 0$.² One can show that the total ghost number u takes on

² Note, however, that J_0^{tot} and e are not defined for twisted states.

integral values, while the two picture numbers π may each be integral (NS) or half-integral (R), corresponding to the four choices of the NSR monodromies. Obviously, $\pi^+ + \pi^- \in \mathbf{Z}$ for the untwisted string. No further restrictions on the values of u or π arise at this point, so that an infinity of (massless) physical vertex operators is anticipated. Like in the $N=1$ string, however, BRST cohomology classes differing by integral values of u or π should correspond to the same physical state, if their spacetime properties agree. Hence, we should identify physical states with equivalence classes of BRST cohomology classes under integral total ghost and picture number changes.

In ref. [2] we have investigated the massless BRST cohomology for various total ghost numbers and generic spacetime momenta in the pictures between $(-2, -2)$ and $(0, 0)$, in the real as well as the holomorphic basis. Given the picture, we found four non-trivial untwisted classes in three ghost sectors, and four non-trivial twisted classes in two ghost sectors. When restricting the lightlike spacetime momentum according to the twisted target spacetime, $k^{3\mu} = 0$, each untwisted class splits into two, a spacetime left-mover ($k^{2+} = 0$, labelled with ‘+’ superscript) and a spacetime right-mover ($k^{2-} = 0$, labelled with ‘-’ superscript). The non-trivial ghost sectors, together with their multiplicity, are given by

$$v \equiv u - \pi^2 - \pi^3 = 1^\pm, \frac{3}{2}^\pm, 2^\pm, 2'^\pm, \frac{5}{2}^\pm, 3^\pm \quad (10a)$$

for the twisted string and by

$$v \equiv u - \pi^+ - \pi^- = 1, 2, 2', 3 \quad (10b)$$

in the untwisted string, where the superscripts distinguish the distinct classes. We conjecture that this pattern persists for all pictures, and that no further physical states appear at massive levels. Then, the twisted string (in real basis) has the following physical spectrum for restricted lightlike spacetime momenta $k^2 \cdot k^2 = 0$, $k^{3\mu} = 0$:

sector	ground states		statistics
(NS,NS)	Φ^+	Φ^-	even
(R,R)	Υ^+	Υ^-	even
(NS,R)	Ξ^+	Ξ^-	odd
(R,NS)	Λ^+	Λ^-	odd

In contrast, the massless and neutral ground states of the untwisted string are a single Φ in the (NS,NS) sector and a single Υ in the (R,R) sector, for generic momenta $k^{i\mu}$ satisfying $k^2 \cdot k^2 + k^3 \cdot k^3 = 0$. The form of the vertex operators depends on the field basis. We shall argue that Φ and Υ are related by spectral flow.

The individual ghost number selection rules for tree-level amplitudes demand that non-vanishing correlation functions have overall total ghost and picture charges of zero and minus two, respectively, and, hence, $v_{\text{total}} = 4$. Of course, spacetime momenta have to add to zero. The non-vanishing two-point functions yield a pairing of vertex operators with

$$(v; \pi^+, \pi^-, k^{\pm\mu}) \longleftrightarrow (4-v; -2-\pi^+, -2-\pi^-; -k^{\pm\mu}) \quad (11)$$

in the holomorphic basis and similarly for the real basis. Setting those two-point functions equal to one normalizes the vertex operators. Clearly, the canonical choice for Φ is the $(-1, -1)$ picture, with

$$V_\Phi = V_{(-1, -1)}^{(1)} = c e^{-\varphi^- - \varphi^+} e^{ik \cdot Z} \quad (12)$$

for $k \cdot k = 0$ and $v = 1$ (indicated as superscript) in the holomorphic basis. For Υ the situation is asymmetric. Here, the canonical vertex operator is

$$\begin{aligned} V_\Upsilon &= V_{(-\frac{3}{2}, -\frac{3}{2})}^{(1)} = c e^{-\frac{3}{2}\varphi^- - \frac{3}{2}\varphi^+} h_\alpha (S^+ S^-)^\alpha e^{ik \cdot Z} \\ \text{or} &= V_{(-\frac{1}{2}, -\frac{1}{2})}^{(1)} = c e^{-\frac{1}{2}\varphi^- - \frac{1}{2}\varphi^+} h_\alpha (\not{k}^+ \not{k}^- S^+ S^-)^\alpha e^{ik \cdot Z}, \end{aligned} \quad (13)$$

with the spin field product

$$(S^+ S^-)^\pm = e^{\pm \frac{1}{2}\phi^+ \pm \frac{1}{2}\phi^-} \quad (14)$$

being an $SO(2, 2)$ Weyl spinor (with $SU(1, 1)$ spinor index $\alpha = \pm$). The polarization spinor h_α satisfying $h_\alpha k^{\alpha\dot{\alpha}} = 0$ ³ represents a single degree of freedom since it must be proportional either to $k^{\alpha+}$ or to $k^{\alpha-}$. The two resulting forms of the vertex, V_Υ^+ and V_Υ^- , are BRST equivalent for generic 2+2 dimensional momenta. In the twisted case the Υ cohomology splits into Υ^+ and Υ^- . It follows that

$$\left\langle V_{(-\frac{3}{2}, -\frac{3}{2})}^{(1)}(k, h) \quad V_{(-\frac{1}{2}, -\frac{1}{2})}^{(3)}(-k, \bar{h}) \right\rangle = h_\alpha (\not{k}^+ \not{k}^-)^{\alpha\beta} \bar{h}_\beta \stackrel{!}{=} 1 \quad (15)$$

which becomes singular when some light-cone projection k^{\pm} vanishes. An extensive list of explicit vertex operators can be found in the appendix of ref. [2].

Fortunately, we do not need to repeat the cohomology analysis for each picture, since an explicit equivalence relation is known. More precisely, the so-called picture-changing operations X^i [20, 18] shift $\pi^j \rightarrow \pi^j + \delta^{ij}$ and $u \rightarrow u + 1$ while commuting

³ The $SO(2, 2)$ vector $k^{i\mu}$ is written as an $SU(1, 1)$ bispinor.

with Q_{BRST} , b_0 and L_0^{tot} , and leaving v unchanged.⁴ Picture-changing in the real basis proceeds as

$$\begin{aligned} V_{(\pi^2+1, \pi^3)} &= [Q_{\text{BRST}}, \xi^2 V_{(\pi^2, \pi^3)}] =: X^2 \cdot V_{(\pi^2, \pi^3)} \\ V_{(\pi^2, \pi^3+1)} &= [Q_{\text{BRST}}, \xi^3 V_{(\pi^2, \pi^3)}] =: X^3 \cdot V_{(\pi^2, \pi^3)} , \end{aligned} \quad (16)$$

where, in the absence of normal ordering between ξ^i and V , one defines the real picture-changing operators

$$X^i(z) := \{Q_{\text{BRST}}, \xi^i(z)\} \quad i = 2, 3 . \quad (17a)$$

Analogously, in the holomorphic basis,

$$X^\pm(z) := \{Q_{\text{BRST}}, \xi^\pm(z)\} \quad (17b)$$

shifts π^+ or π^- by one unit, while leaving $v = u - \pi^+ - \pi^-$ untouched.

By construction, the X are BRST invariant but *not* BRST trivial due to the lack of the zero modes of ξ in the bosonization formulae [18]. The picture-changing operators just introduced take the following explicit form:

$$\begin{aligned} X^2 &= c\partial\xi^2 + e^{\varphi^2} \left[G + \frac{i}{2}\tilde{c}\beta^3 + 2i(\partial\tilde{b})\gamma^3 + 4i\tilde{b}\partial\gamma^3 \right] + 2e^{2\varphi^2}b\partial\eta^2 + \partial(e^{2\varphi^2}b)\eta^2 , \\ X^3 &= c\partial\xi^3 + e^{\varphi^3} \left[\overline{G} - \frac{i}{2}\tilde{c}\beta^2 - 2i(\partial\tilde{b})\gamma^2 - 4i\tilde{b}\partial\gamma^2 \right] + 2e^{2\varphi^3}b\partial\eta^3 + \partial(e^{2\varphi^3}b)\eta^3 , \end{aligned} \quad (18a)$$

$$\begin{aligned} X^+ &= c\partial\xi^+ + e^{\varphi^-} \left[G^+ + 2\partial\tilde{b}\gamma^+ + 4\tilde{b}\partial\gamma^+ - 2b\gamma^+ \right] , \\ X^- &= c\partial\xi^- + e^{\varphi^+} \left[G^- - 2\partial\tilde{b}\gamma^- - 4\tilde{b}\partial\gamma^- - 2b\gamma^- \right] . \end{aligned} \quad (18b)$$

It is clear that X^\pm are not just linear combinations of X^i . The two types of picture-changing operators differ in two respects. The holomorphic version does not contain $e^{2\varphi}$ terms, and it also lacks any \tilde{c} dependence. The latter means that X^\pm commute with J_0^{tot} , whereas X^i do so only modulo BRST-exact terms.

Since picture-changing X establishes an equivalence of cohomology classes, its inverse Y can only be well-defined modulo BRST-trivial terms and may, like X itself, possess a BRST-trivial nonzero kernel. We require in the real form that

$$[Q_{\text{BRST}}, Y^i] = 0 \quad \text{and} \quad Y^2(z) X^2(w) \sim 1 \sim Y^3(z) X^3(w) \quad (19)$$

but do not constrain the mixed products. The quantum numbers of Y^i are determined as $(h, u, \pi^j) = (0, -1, -\delta^{ij})$. A simple ghost number analysis shows that this leaves

⁴ The modified ghost number v was conveniently introduced for this reason.

only a single candidate for each φ^i ghost charge value below -1 . In the case of Y^2 , for instance, we are forced to write a linear combination of

$$Y_k^2 = c \left(\gamma^3 \right)^{k-2} \partial^{k-1} \xi^2 \dots \partial^2 \xi^2 \partial \xi^2 e^{-k\varphi^2} \quad k \geq 2 \quad (20)$$

which satisfy

$$Y_k^2(z) X^2(w) \sim \delta_{k2} + O(z-w) . \quad (21)$$

The promising first term,

$$Y_2^2 = c \partial \xi^2 e^{-2\varphi^2} , \quad (22)$$

is identical with the inverse picture-changing operator of the $N=1$ string, but fails to be BRST invariant for the $N=2$ string. Astoundingly, this failure can be corrected by adding an infinite series of Y_k^2 , with $k = 4, 6, 8, \dots$. In other words, the coefficients in

$$Y^2 = \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \left(\prod_{\ell=1}^{k-1} \ell! \right)^{-1} Y_k^2 = Y_2^2 + \frac{1}{12} Y_4^2 + \frac{1}{34560} Y_6^2 + \dots \quad (23)$$

lead to a chain of cancellations among BRST commutators of successive terms. With some effort, the formal series can be summed to the non-local expression

$$Y^2(w) = \sin \oint_w [\gamma^2 \beta^3 - \gamma^3 \beta^2] \cdot Y_1^2(w) = -i \sinh(2 \operatorname{ad} J_0^{\text{tot}}) \cdot Y_1^2(w) \quad (24)$$

where we introduced

$$Y_1^2 = -c \xi^3 e^{-\varphi^3 - \varphi^2} = c(\gamma^3)^{-1} \delta(\gamma^2) \quad (25)$$

and understand the action on Y_1^2 as a power series of iterated commutators. Of course, a mirror image expression emerges for Y^3 . Note that the Y^i are pure ghost operators and do not contain any matter fields. A similar analysis in the holomorphic basis fails to produce any candidate for Y^\pm . Still, we suspect that some, necessarily non-local, inverse picture-changing operators exist in this case as well.

So far, our treatment of local vertex operators has not been systematic. Now we are going to employ a unified formalism which simultaneously deals with all chiral vertex operators defined in arbitrary pictures. For the purpose of mutual locality of vertex operators, we may temporarily forget about their momentum-dependence⁵ and concentrate on their ghost, spin, and twist field structure. In operator products, the (b, c) and matter twist fields never lead to branch cuts, since t^3 always occurs in

⁵ For the tree-level correlation functions, their momentum dependence is essentially absorbed into the usual Koba-Nielsen factor. In the twisted sector, the constrained kinematics (only k^{2-} non-zero) allows merely Z^{2+} which does not get twisted.

combination with a ghost twist field, i.e. $t_+^3 = t^3 e^{\tilde{\sigma}/2}$, and those have meromorphic OPE with one another. Only $\psi^{i\pm}$ and $S^{i\pm}$ as well as (β^i, γ^i) and their spin fields may ruin locality. After real bosonization, any vertex operator is a linear combination of terms proportional to ⁶

$$\exp \left[p_2 \phi^2 + q_2 \varphi^2 + p_3 \phi^3 + q_3 \varphi^3 \right] , \quad (26)$$

where p_i and q_i take integral or half-integral values.

More specifically, each pair $[p_i, q_i]$ belongs to a lattice

$$\Gamma_w^{(i)} = \Pi^{1,1} \equiv \mathbf{Z}^{1,1} \cup \left[\left(\frac{1}{2}, \frac{1}{2} \right) + \mathbf{Z}^{1,1} \right] = \underbrace{(o) \cup (v)}_{\text{NS}} \cup \underbrace{(s) \cup (c)}_{\text{R}} \quad (27)$$

which may be regarded as a half-integral lorentzian weight lattice. Here, the scalar product has been chosen as $[p, q] \cdot [p', q'] = pp' - qq'$. Like for $so(2n)$, the weight lattice decomposes into the root lattice (o) and three copies of it, each shifted by a different elementary weight vector and conventionally denoted by (v) , (s) and (c) . The lorentzian length-squared $p^2 - q^2$ is even integer except for weights in (v) where it is odd. On the other hand, the contribution to the conformal dimension $h = \frac{1}{2} \sum_i (p_i^2 - q_i^2 - 2q_i)$ of the operator (26) is integral for (o) and half-integral for (v) , (s) and (c) .

For our purposes, we have to consider the combined weights $[p_2, q_2; p_3, q_3]$ which form the still half-integral 2+2 dimensional weight lattice

$$\Gamma_w = \Gamma_w^{(2)} \oplus \Gamma_w^{(3)} = \{(r_2; r_3)\} . \quad (28)$$

From the 16 conjugacy classes $(r_2; r_3)$ those 6 containing a single (o) contribute half-integrally to h and, hence, do not contain physical states. The remaining 10 classes Γ'_w do not form a lattice. They split into 6 *even* (= untwisted) and 4 *odd* (= twisted) classes:

class	(o, o)	(v, v)	(s, s)	(c, c)	(s, c)	(c, s)	(v, s)	(v, c)	(s, v)	(c, v)
state	—	Φ^\pm	Υ^-	Υ^+	—	—	Ξ^-	Ξ^+	Λ^-	Λ^+

Like in the $N=1$ string, we should like to identify commuting (spacetime bosonic) vertex operators with *even untwisted* weights and anticommuting (spacetime fermionic) vertex operators with *odd twisted* weights. The lattice consideration takes us out of the ‘canonical pictures’ because the fusion algebra only closes in the infinite set of all

⁶ Eventual derivatives of ϕ^i or φ^i are irrelevant again.

pictures. Since the picture numbers π^i agree *modulo 1* with q_i , integral (half-integral) values of $\pi^2 + \pi^3$ coincide with even untwisted bosonic (odd twisted fermionic) vertex operators, and we may use these terms interchangeably.

The relevance of the lattice description derives from the basic OPE of two bosonized operators as in eq. (26),

$$\exp[p_i\phi^i + q_i\varphi^i](z) \exp[p'_i\phi^i + q'_i\varphi^i](w) \sim (z-w)^{p_i p'_i - q_i q'_i} \exp[(p_i + p'_i)\phi^i + (q_i + q'_i)\varphi^i](w) , \quad (29)$$

which relates the mutual locality of two vertex operators to the integrality of their weight's lorentzian scalar product. Moreover, fusion simply corresponds to adding weights. Our task is to enumerate all local vertex operator subalgebras. Apparently, this amounts to classifying the *integral sublattices* $\Gamma_{\text{int}} \subset \Gamma'_w$. Each such Γ_{int} is obtained from Γ_w by some GSO projection [21] and leads to a different string model. By inspection, one finds six maximal possibilities,

$$\begin{aligned} I & (o; o) \cup (v; v) \cup (s; s) \cup (c; c) \\ II & (o; o) \cup (v; v) \cup (s; c) \cup (c; s) \\ III & (o; o) \cup (s; s) \cup (v; c) \cup (c; v) \\ IV & (o; o) \cup (c; c) \cup (v; s) \cup (s; v) \\ V & (o; o) \cup (s; c) \cup (v; s) \cup (c; v) \\ VI & (o; o) \cup (c; s) \cup (v; c) \cup (s; v) . \end{aligned} \quad (30)$$

By construction, these lattices are self-dual, which is expected to be crucial for modular invariance. It is non-trivial that these projections are compatible with picture-changing, since $X^i \in (o, o) \cup (v, v)$ and $Y^i \in (o, o)$. Helicity flips $(s) \leftrightarrow (c)$ connect I and II as well as III through VI . Hence, there are only *two* types of essentially distinct GSO projections, say, model I , with four bosonic classes, and model III , with two bosonic and two fermionic ones. We may call them ‘notwist’ and ‘twisted’, respectively. Clearly, the surviving physical states are

$$\begin{aligned} \text{GSO}_{\text{notwist}} & \implies (\Phi, \Upsilon) , \\ \text{GSO}_{\text{twisted}} & \implies (\Upsilon^-, \Xi^+, \Lambda^+) . \end{aligned} \quad (31)$$

From the analysis of the type III GSO projection in eq. (30), we should expect a *second* bosonic state in $(o; o)$ in addition to $\Upsilon^- \in (s; s)$. Using picture equivalence, this new state must be represented in the $(-1, -1)$ picture. However, dimensional analysis easily shows that there can be no massless $(o; o)$ state in this picture; the only massless state at all is $\Phi \in (v; v)$! We must conclude that for the twisted theory the

$(o; o)$ class has no BRST cohomology, i.e. contains only trivial states. Unfortunately, there is no match between bosons and fermions.

Let us take a closer look at the twisted model since it holds the promise of space-time supersymmetry. The twisted spacetime momentum constraint $k^{3\mu} = 0$ implies for lightlike momenta that the two fermions are left-movers ($k^{2+} = 0$) while the boson is a right-mover ($k^{2-} = 0$). Effectively, the target space dimensionality has been reduced to $D = 1+1$, excluding interactions between right- and left-movers. Indeed, it is not hard to check that the fusion rules are trivial, i.e. any two fields fuse to a BRST trivial state. Consequently, the only allowed three-point function vanishes, $\langle \Upsilon^- \Xi^+ \Lambda^+ \rangle = 0$. The prospective spacetime supersymmetry generators,

$$Q^2 = \oint b_{-1} V_{\Lambda^+}(k=0) \quad Q^3 = \oint b_{-1} V_{\Xi^+}(k=0) \quad , \quad (32)$$

satisfy

$$\{Q^i, Q^j\} = 0 \quad i, j = 2, 3 \quad . \quad (33)$$

They qualify as exterior derivatives rather than supersymmetry charges.

The untwisted string, in contrast, permits interactions. Its fusion rules read

$$[\Phi] \cdot [\Phi] = [\Phi] \quad , \quad [\Phi] \cdot [\Upsilon] = [\Upsilon] \quad , \quad [\Upsilon] \cdot [\Upsilon] = [\Phi] \quad , \quad (34)$$

and the tree-level three-point functions become

$$\langle V_\Phi(k_1) V_\Phi(k_2) V_\Phi(k_3) \rangle = \langle V_\Phi(k_1) V_\Upsilon(k_2) V_\Upsilon(k_3) \rangle = \epsilon^{ij} k_2^i \cdot k_3^j \quad , \quad (35)$$

the only non-zero bilinear $U(1, 1)$ invariant. All possible four-point functions have been checked to vanish identically, due to non-trivial kinematic identities in 2+2 dimensions [9]. It has been conjectured that all higher-point functions and loop correlators vanish as well. If this is correct, the effective spacetime action for the Φ degree of freedom will be that of self-dual Yang-Mills for the open-string case or of self-dual gravity for the closed-string case, in 2+2 dimensions.

Like in the $N=1$ superstring, picture-changing operators have singular OPE with one another,

$$X^i(z) X^j(w) \sim (z-w)^{-2} \{Q_{\text{BRST}}, \Delta^{ij}\} \quad i, j = 2, 3 \quad , \quad (36)$$

which spoils the gauge invariance of Witten's superstring field theory with the NS string field in the -1 picture [22, 23, 24]. In the holomorphic basis of the $N=2$ string, however, *like* picture-changing operators show a *regular* OPE. Thus, we can follow the recipe given in ref. [25] and use left-moving X^+ s and right-moving X^- s

exclusively, which avoids singular X collisions anywhere in the moduli space. We conclude that the ‘canonical’ $N=2$ string field theory is well-behaved and does not suffer from singular contact terms.

Let me close with a few remarks on the structure of loop amplitudes. There have been some preliminary one-loop computations [17, 26] but it is fair to say that the issue is still unclear. A closed $N=2$ super Riemann surface is characterized by its genus $g \in \mathbf{N}$ and its instanton or Chern number $c \in \mathbf{Z}$ as well as (in complex count)

- $3g-3$ metric moduli,
- $2g-2+c$ positively charged fermionic moduli,
- $2g-2-c$ negatively charged fermionic moduli,
- g abelian gauge or $U(1)$ moduli,

for $g > 1$. The string measure supports only a non-negative number of fermionic moduli, which restricts the sum over c for any given g .

The metric moduli are well studied, and the fermionic moduli will have to be integrated out formally around split surfaces. The abelian gauge moduli are a new feature of the $N=2$ string. The $U(1)$ moduli space is the space of flat $U(1)$ connections which is nothing but the Jacobian torus $J = \mathbf{C}^g / (\mathbf{Z}^g + \Omega \mathbf{Z}^g)$. Since any flat $U(1)$ connection A is characterized by the $2g$ phases $\exp \oint A$ around the homology cycles, there is a one-to-one correspondence between J and the complex torus $Pic = \mathbf{C}^g / \mathbf{Z}^{2g}$ which describes the space of holomorphic line bundles with a given degree. A point in Pic determines the twists on the homology, i.e. the constant monodromy phases which are the transition functions around the cycles. The spin structures labelling holomorphic spinor bundles sit on the half-points $(\frac{1}{2}\mathbf{Z}/\mathbf{Z})^{2g}$ of Pic . Now observe that by a unitary transformation via

$$U(P) = \exp \left\{ \int_{P_0}^P A \right\} \quad (37)$$

on a given spinor bundle we change the monodromies. With a suitable $A \in J$ we can reach any point in Pic and, in particular, move to any other spin structure. The NSR fermions couple to the abelian gauge field as in eq. (37). Hence, the sum over NSR spin structures is automatically contained in the integration over the $U(1)$ moduli. In fact, any change in NSR monodromies (jointly for all $\psi^{i\mu}$) can be traded for a shift in $U(1)$ moduli space and, hence, cannot be physical. This feature is not restricted to $g \geq 1$ but appears just as well for the n -punctured sphere, i.e. in

tree-level amplitudes. Here, any (R,R) puncture can be turned into an (NS,NS) one, since there are $n-1$ independent cycles and the sum of all twists has to vanish. As a result, Φ and Υ states cannot be physically distinguished. This is consistent with our observation that their correlators coincide. However, at present we do not know an explicit spectral flow operator implementing the $\Phi \leftrightarrow \Upsilon$ interchange on the level of the vertex operators.

When the fermionic moduli are formally integrated out with a simple choice of fermionic beltrami differentials, the appropriate number of picture-changing operators appear in the string measure. Furthermore, we have to do the $U(1)$ moduli integration, which adds to the measure g insertions of a novel ‘ $U(1)$ projection operator’ containing \tilde{b} and J_{tot} . Its significance remains to be clarified. Together with the usual $3g-3$ insertions of b ghosts folded with regular beltrami differentials and a single \tilde{c} this completes the measure for the remaining integration over the metric moduli of standard Riemann surfaces.

Note added in 12/94:

N. Berkovits and C. Vafa pointed out to us the relevance of their recent work [27, 28] where they imbed the $N=2$ string into a new $N=4$ topological string theory. This allows them to rewrite the critical $N=2$ string n -point functions as correlators in the topological theory, where their vanishing to all string loop orders can be proved, with the known exceptions. I would also like to mention a new preprint by H. Lü and C. Pope [29] which overlaps with our results but takes a different viewpoint.

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